JOURNAL OF APPROXIMATION THEORY 63, 3-29 (1990)

K-Functionals and Weighted Moduli of Smoothness

Z. DITZIAN*

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

AND

V. TOTIK†

Bolyai Institute, Attila József University, Aradi V. tere 1, Szeged 6720, Hungary

Communicated by P. L. Butzer

Received March 22, 1988; revised August 29, 1988

1. INTRODUCTION

Characterization of K-functionals is an important aspect of the theory of interpolation spaces. For the couple L_p and $W_p^r(\varphi^r)$, we found in [4] a new kind of modulus of smoothness that is equivalent to the K-functional. This has had many applications in approximation theory (see [4, Part II]). In particular, the K-functional is related to the characterization of the rate of approximation given by different approximation processes. Earlier attempts in this direction (see, e.g., [1, 2]) suggested the use of ordinary weighted moduli of smoothness. However, as it has been shown, this is possible only for $p = \infty$ (for $p = \infty$ see [3, p. 322]). In the present work, we pursue further this line of investigation. Being aware of counterexamples in [6] which show that such equivalence is not possible for L_p, $1 \leqslant p < \infty$, we show here that the *characterization* of the K-functional of a function f is possible *by some weighted ordinary modulus of smoothness of a related function* (not f itself). In the last section we give several concrete applications. This demonstrates how our theorems characterizing the K-functionals are used to determine the rate of approximations for various approximation processes.

* Supported by NSERC of Canada, Grant A4816.

† Supported by the Hungarian National Science Foundation for Research, Nos. 1157, 1801.

0021-9045/90 $3.00

We now describe the problem and solution in more specific terms. The K-functional between $L_p(a, b)$ and $W_p^r(\varphi^r)$ (weighted Sobolev space with weight φ^r) is given by

$$K_{r,\varphi}(f, t^r)_p = \inf_{g^{(r-1)} \in AC_{\text{loc}}} (\|f - g\|_p + t^r \|\varphi^r g^{(r)}\|_p). \tag{1.1}$$

We proved in [4, Theorem 2.1.1] that, under suitable conditions on φ (cf. [4, Sect. 1.2]), the modulus of smoothness,

$$\omega_\varphi^r(f, t)_p = \sup_{0 < h \leqslant t} \|\Delta_{h\varphi}^r f\|_p, \tag{1.2}$$

is equivalent to the K-functional in (1.1), i.e.,

$$K_{r,\varphi}(f, t^r)_p \sim \omega_\varphi^r(f, t)_p \qquad (t \to 0+). \tag{1.3}$$

Note that in (1.2), the increment $h\varphi(x)$ changes as x varies through (a, b) while our aim here is to find another expression of smoothness (of a related function) which is formed by constant increments (i.e., for which the "step weight" is 1) and which is still useable to describe $K_{r,\varphi}(f, t^r)_p$. We shall show that this is possible under some natural assumptions. A complete characterization of the important case when the K-functional satisfies

$$K_{r,\varphi}(f, t^r)_p = O(t^\alpha) \qquad \text{for} \quad t \to 0$$

will be given.

Section 2 contains the notations and our terminology. In Section 3, we state our main results, the proofs of which will be given in Section 6. Section 5 is devoted to some preliminaries to these proofs. In Section 4, we discuss an example which shows that our theorems are sharp.

Finally, in the last section, we synthesize our results with earlier ones to derive theorems on polynomial and operator approximation. It should be noted that these results, which are applications of the main theorems, are of utmost importance here. They are particularly essential because of the technical nature of the main theorems of this paper. On the one hand, these applications show the generality and nontriviality of our results by providing rather surprising equivalences between approximation errors and some (strangely related) smoothness conditions. On the other hand, the failure of the above equivalences for L_p with $p > 2$ (while valid for $p < 2$) perhaps explains why these results were not discovered previously and underlines the necessity of the new measure of smoothness given in [4] that provides equivalences for all L_p spaces in a uniform way.

2. NOTATIONS AND TERMINOLOGY

The weight functions w on $(a, b) \subseteq R$ discussed in this paper are positive measurable functions on the interval $(a, b) \subseteq R$ which are bounded away from zero and infinity on every compact subinterval of (a, b). Since, by linear substitution, we can carry (a, b) into one of the intervals $(0, 1)$, $(0, \infty)$, or $(-\infty, \infty)$, we can always assume that (a, b) is one of these.

Let $f \sim g$ mean that $1/C \leqslant f/g \leqslant C$ for some constant C in the range considered. For example, "$f(x) \sim g(y)$ if $x \sim y$" means that if $1/C \leqslant x/y \leqslant C$, then $1/C_1 \leqslant f(x)/g(y) \leqslant C_1$ for some C_1.

We shall consider L_p spaces and assume throughout the paper that

$$1 \leqslant p < \infty. \tag{2.1}$$

For a weight function φ and a positive integer r, the weighted Sobolev space $W_p^r(\varphi^r)$ is given by

$$W_p^r(\varphi^r) = \{f \mid f^{(r-1)} \in AC_{\text{loc}}, \varphi^r f^{(r)} \in L_p\},$$

where $f^{(r-1)} \in AC_{\text{loc}}$ means that f is $(r-1)$-times differentiable and its $(r-1)$st derivative is locally absolutely continuous; i.e., it is absolutely continuous on every compact subinterval of (a, b).

Even if we restrict our interest only to the K-functionals of the form (1.1), in our considerations, weighted L_p spaces inevitably appear (see [4, Chap. 6]). Therefore, we might as well introduce an additional weight w and the weighted K-functional $K_{r,\varphi}(f, t^r)_{w,p}$ given by

$$K_{r,\varphi}(f, t^r)_{w,p} = \inf_{g^{(r-1)} \in AC_{\text{loc}}} (\|w(f-g)\| + t^r \|w\varphi^r g^{(r)}\|_p). \tag{2.2}$$

The corresponding weighted analogue of (1.2) is defined in a slightly more complicated way and we shall introduce it after having said something about φ and w.

We will not give the exact general conditions on φ that ensure (1.3). These can be found in [4, Sect. 1.2]. For our purposes it is enough to make the following natural assumptions. We assume that there exist two numbers $\beta(a)$ and $\beta(b)$ such that $\beta(c) \geqslant 0$ if c (where c is equal to a or b) is a finite endpoint of (a, b), and $\beta(c) \leqslant 1$ if c is infinite. We further assume that when $(a, b) = (0, 1)$, $(0, \infty)$, or $(-\infty, \infty)$,

$$\varphi(x) \sim \begin{cases} |x|^{\beta(a)} & \text{as} \quad x \to a+0 \quad (a = 0 \text{ or } a = -\infty) \\ x^{\beta(\infty)} & \text{as} \quad x \to \infty \qquad (b = \infty) \\ (1-x)^{\beta(1)} & \text{as} \quad x \to 1-0 \quad (b = 1). \end{cases} \tag{2.3}$$

Since $\varphi \sim \psi$ implies $K_{r,\varphi} \sim K_{r,\psi}$, we can assume (and this is the second assumption made throughout the paper, the first being (2.1)) that $\varphi \in C^r(a, b)$ and in certain neighbourhoods of a and b the equivalence "$\sim$" becomes equality in (2.3). For example, if $(a, b) = (0, \infty)$, then we assume that

$$\varphi(x) = x^{\beta(0)} \qquad \text{for} \quad 0 < x < 1$$

and

$$\varphi(x) = x^{\beta(\infty)} \qquad \text{for} \quad x > 2.$$

To discuss the K-functional $K_{r,\varphi}(f, t^r)_{w,p}$, we will make the following assumption about the weight w. There exist two numbers $\gamma(a)$ and $\gamma(b)$ such that

$$\gamma(c) \geqslant 0 \qquad \text{if} \quad c \ (c = a \text{ or } c = b) \text{ is a finite endpoint with } 0 \leqslant \beta(c) < 1$$

and

$$w(x) \sim \begin{cases} |x|^{\gamma(a)} & \text{as} \quad x \to a+ \quad (a = 0 \text{ or } a = -\infty) \\ x^{\gamma(\infty)} & \text{as} \quad x \to \infty \quad (b = \infty) \\ (1-x)^{\gamma(1)} & \text{as} \quad x \to 1- \quad (b = 1). \end{cases} \tag{2.4}$$

Note that $\gamma(c)$ is not restricted if c is not finite or if $\beta(c) \geqslant 1$. These assumptions are satisfied by most weight functions φ and w appearing in applications (see Section 7 below).

The symmetric rth difference is given by

$$\Delta_h^r f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(x + (rh/2) - kh)$$

and the forward rth difference by

$$\vec{\Delta}_h^r f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(x + (r-k)h).$$

When the expression $w\Delta_{h\varphi}^r f$ appears in the norm $L_p(a, b)$, it will be assumed that $\Delta_{h\varphi}^r f = 0$ if $(x - rh\varphi(x)/2, x + rh\varphi(x)/2) \not\subseteq (a, b)$.

For simplicity, we define the weighted analogue of (1.2) only for $(a, b) = (0, \infty)$. In this case, we overcome the difficulties near both finite and infinite endpoints and, therefore, the corresponding definitions and proofs for $(a, b) = (0, 1)$ and $(a, b) = (-\infty, \infty)$ should not cause any problem (cf. [4, Sect. 6.1 and Appendix B] and the discussion at the end of the proof of

Theorem 1). Thus, let $(a, b) = (0, \infty)$ and when $\beta(0) \geqslant 1$ or $\gamma(0) = 0$, we simply write

$$\omega_\varphi^r(f, t)_{w,p} = \sup_{0<h\leqslant t} \|w\Delta_{h\varphi}^r f\|_p.$$

In the remaining case, that is, when $0 \leqslant \beta(0) < 1$ and $\gamma(0) > 0$, we define

$$\omega_\varphi^r(f, t)_{w,p} = \sup_{0<h\leqslant t} \|w\Delta_{h\varphi}^r f\|_{L_p(t^*, \infty)} + \sup_{0<h\leqslant t} \|w\vec{\Delta}_{h\varphi}^r f\|_{L_p(0, 12t^*)},$$

where (for $t < 1$)

$$t^* = (rt)^{1/(1-\beta(0))}. \tag{2.5}$$

With these notations, we have [4, Theorem 6.1.1]

$$K_{r,\varphi}(f, t^r)_{w,p} \sim \omega_\varphi^r(f, t)_{w,p} \qquad (t \to 0). \tag{2.6}$$

We will use the notation

$$A \ll B$$

extensively rather than the expression "there exists a constant c such that $|A| \leqslant c\,|B|$."

3. Main Results

Let $\Gamma(x)$ be given by

$$\Gamma(x) = \int_{1/2}^{x} \frac{dt}{\varphi(t)}$$

and let

$$\theta = \Gamma^{-1} \tag{3.1}$$

be the inverse function of Γ. We observe that Γ maps (a, b) (where (a, b) is $= (0, 1)$, $(0, \infty)$, or $(-\infty, \infty)$) onto an interval (A, B) and θ maps (A, B) onto (a, b). With this notation, we will prove the following result.

THEOREM 1. *Assume* $1 \leqslant p < \infty$ *and*

$$\gamma(c) + (1-r)(1-\beta(c)) > -1/p \qquad \textit{if} \quad c\ (c = a \textit{ or } c = b)$$
$$\textit{is a finite endpoint of } (a, b) \textit{ with } 0 \leqslant \beta(c) < 1. \tag{3.2}$$

For $F = f \circ \theta$ *where* θ *is defined by* (3.1), *both*

$$\omega_1^r(F, t)_{(w\circ\theta)(\varphi\circ\theta)^{1/p}, p} \ll \omega_\varphi^r(f, t)_{w,p} + t^r \|wf\|_p, \tag{3.3}$$

and

$$\omega_\varphi^r(f, t)_{w,p} \leqslant \omega_1^r(F, t)_{(w\circ\theta)(\varphi\circ\theta)^{1/p}, p} + t^r \|(w\circ\theta)(\varphi\circ\theta)^{1/p} F\|_p \tag{3.4}$$

hold.

This gives the type of comparison mentioned in the Introduction since in the definition of $\omega_1^r(f, t)_{W,p}$, the increment h is constant; i.e., the "step weight" is identically equal to 1, $\omega_1^r(F, t)_{W,p}$ is an ordinary weighted modulus of smoothness.

Remarks. 1. Note that $\omega_\varphi^r(f, t)_{w,p}$ is defined on (a, b) while $\omega_1^r(F, t)_{(w\circ\theta)(\varphi\circ\theta)^{1/p}, p}$ is defined on (A, B).

2. The result holds for many other Γ's and θ's as well. For example, if $(a, b) = (0, \infty)$ and $\theta\colon (0, \infty) \to (0, \infty)$ satisfies $\theta \in C^r(0, \infty)$, $\theta' \sim \varphi\circ\theta$, and for $1 \leqslant i \leqslant r$

$$\theta^{(i)}(x) \leqslant \theta(x) \cdot x^{-i},$$

then this θ can replace the θ given in (3.1).

3. Note also that if χ is a linear mapping between (A', B') and (A, B), then Theorem 1 holds for θ if and only if it is true for $\Theta = \theta\circ\chi$ (cf. (2.6)).

We shall use these remarks in the proofs and in our applications.

As the value of r becomes larger, the condition (3.2) becomes more severe. Therefore, it is important to give comparisons under the following less restrictive assumptions.

THEOREM 2. *Suppose* $1 \leqslant p < \infty$, $F = f\circ\theta$, $0 < s < r$ *for the integers* s *and* r, *and*

$$s + \gamma(c) + (1 - r)(1 - \beta(c)) > -1/p \quad \text{if} \quad c\ (c = a \text{ or } c = b) \text{ is a finite endpoint with } 0 \leqslant \beta(c) < 1. \tag{3.5}$$

Then the assumption

$$I(f) = \int_0^1 (\omega_\varphi^r(f, u)_{w,p}/u^{s+1})\, du < \infty \tag{3.6}$$

implies $f^{(s-1)} \in AC_{\mathrm{loc}}$ *and*

$$\omega_1^r(F, t)_{(w\circ\theta)(\varphi\circ\theta)^{s+1/p}, p} \leqslant t^s \int_0^t (\omega_\varphi^r(f, u)_{w,p}/u^{s+1})\, du + t^r(I(f) + \|wf\|_p). \tag{3.7}$$

Conversely (where (3.5) *still holds), the assumption*

$$\mathscr{I}(f)=\int_0^1 (\omega_1^r(F,u)_{(w\circ\theta)(\varphi\circ\theta)^{s+1-r},p}/u^{s+1})\,du<\infty$$

implies $F^{(s-1)}\in AC_{\mathrm{loc}}$ *and*

$$\omega_\varphi^r(f,t)_{w,p}\ll t^s\int_0^t (\omega_1^r(F,u)_{(w\circ\theta)(\varphi\circ\theta)^{s+1-r},p}/u^{s+1})\,du$$
$$+\,t^r(\mathscr{I}(f)+\|(w\circ\theta)(\varphi\circ\theta)^{s+1-r}F\|_p). \tag{3.8}$$

Observe that we have

$$t^s\int_0^t (\omega_\varphi^r(f,u)_{w,p}/u^{s+1})\,du\gg t^s\int_0^t u^{r-s-1}\,du\gg t^r,$$

i.e., the first term on the right of (3.7) or (3.8) is the significant term. We also mention that for $s=r-1$, (3.5) is always valid.

Finally, we turn to the characterization of

$$\omega_\varphi^r(f,t)_{w,p}=\mathcal{O}(t^\alpha)$$

and, hence, that of $K_{r,\varphi}(f,t)_{w,p}=\mathcal{O}(t^\alpha)$.

THEOREM 3. *Suppose* $1\le p<\infty$, $wf\in L_p(a,b)$, $0<\alpha<r$, *and* $s=[\alpha]$ *if* α *is not an integer and* $s=\alpha-1$ *if it is. Then for* $F^{(s)}=f^{(s)}\circ\theta$,

$$\omega_\varphi^r(f,t)_{w,p}=\mathcal{O}(t^\alpha) \tag{3.9}$$

implies that $f^{(s-1)}\in AC_{\mathrm{loc}}$ *and*

$$\omega_1^r(F,t)_{(w\circ\theta)(\varphi\circ\theta)^{s-1-r},p}=\mathcal{O}(t^\alpha). \tag{3.10}$$

Conversely, for $f=F\circ\theta^{-1}$, (3.10) *implies that* $F^{(s-1)}\in AC_{\mathrm{loc}}$ *and for* $f^{(s)}=F^{(s)}\circ\theta^{-1}$ (3.9) *holds.*

COROLLARY 1. *Under the assumptions of Theorem* 3

$$K_{r,\varphi}(f,t)_{w,p}=\mathcal{O}(t^\alpha)$$

and (3.10) *are equivalent.*

4. AN EXAMPLE

In this section, we show that the results in the preceding one are sharp. First of all, notice that the term $t^r\,\|wf\|_p$ on the right of (3.3) is needed even if its order of magnitude is usually smaller than that of $\omega_\varphi^r(f, t)_{w,p}$. In fact, if f is a polynomial of degree at most $(r-1)$, then $\omega_\varphi^r(f, t)_{w,p} \equiv 0$ while

$$\omega_1^r(f \circ \theta, t)_{W,p} \not\equiv 0 \qquad \text{for} \quad r > 1.$$

Recall that for any f which is not a polynomial of degree at most $r-1$ and for any weight function W,

$$\liminf_{t \to 0+} \omega_\varphi^r(f, t)_{w,p}/t^r > 0$$

(cf. [4, Sect. 4.2] or (2.6)).

With regard to the range of parameters in Theorem 1, we have:

ASSERTION 1. *Suppose* $0 \leqslant \beta < 1$, $\varphi(x) = x^\beta$, $w(x) = x^\gamma$, $\gamma \geqslant 0$, $1 \leqslant p < \infty$, *and*

$$\gamma + (1-r)(1-\beta) \leqslant -1/p, \tag{4.1}$$

(*i.e., assume that* (3.2) *does not hold*). *Then the conclusion of Theorem* 1 *fails.*

Proof. Let $f \in C^r(0, \infty)$ such that

$$f(x) = \begin{cases} (1-\beta)\, x^{1-\beta} \log x - x^{1-\beta} & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x > 2. \end{cases}$$

We can choose (see Remark 3 of Section 3)

$$\Gamma(x) = (1-\beta) \int_0^x u^{-\beta}\, du = x^{1-\beta}$$

and

$$\theta(x) = x^{1/(1-\beta)}.$$

Then

$$(f \circ \theta)(x) = x \log x - x$$

and

$$(f \circ \theta)'(x) = \log x.$$

Hence, we can write (see also [4, Sects. 3.4 and 8.5] and (5.7) below)

$$
\begin{aligned}
\omega_1^r(f \circ \theta, t)_{(w \circ \theta)(\varphi \circ \theta)^{1/p}, p}
&\sim t^r \left\{ \int_{rt}^1 |(w \circ \theta)(u)((\varphi \circ \theta)(u))^{1/p} (f \circ \theta)^{(r)}(u)|^p \, du \right\}^{1/p} \\
&\sim t^r \left\{ \int_{rt}^1 u^{((\gamma + \beta/p)/(1-\beta) + 1 - r)p} \, du \right\}^{1/p} \\
&\sim \begin{cases} t^r |\log t|^{1/p} & \text{if equality holds in (4.1)} \\ t^{1 + (\gamma + 1/p)/(1-\beta)} & \text{if strict inequality holds in (4.1).} \end{cases}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\omega_\varphi^r(f, t)_{w, p} &\sim t^r \left\{ \int_{(rt)^{1/(1-\beta)}}^1 |w(u)\, \varphi^r(u)\, f^{(r)}(u)|^p \, du \right\}^{1/p} \\
&\sim t^r \left\{ \int_{(rt)^{1/(1-\beta)}}^1 u^{(\gamma + r\beta + 1 - \beta - r)p} |\log u|^p \, du \right\}^{1/p} \\
&\sim \begin{cases} t^r |\log t|^{1 + 1/p} & \text{if equality holds in (4.1)} \\ t^{1 + (\gamma + 1/p)(1-\beta)} |\log t| & \text{if inequality holds in (4.1).} \end{cases}
\end{aligned}
$$

These two moduli have different orders. ∎

5. Preliminaries to the Proofs

We first quote from [4, Theorem 6.11] the following result.

THEOREM A. *Let* φ, w, p, *and* r *satisfy* (2.3), (2.4), *and* (3.2). *Then*

$$
K_{r,\varphi}(f, t^r)_{w,p} \sim \omega_\varphi^r(f, t)_{w,p} \qquad (t \to 0+). \tag{5.1}
$$

It will be important to observe:

THEOREM B. *The equivalence* (5.1) *is also valid if* $(a, b) = (-\infty, \infty)$, $\varphi \equiv 1$, *and*

$$
w(x) = W(e^x), \tag{5.2}
$$

where W *is a weight function satisfying condition* (2.4) *on* $(0, \infty)$.

Since (5.2) implies that $w(x) \sim w(y)$ if $x - y \sim 1$ when $x \to \pm\infty$, the proof of Theorem B is an easy modification of the proof of Theorem A in [4].

Note that (5.1) implies that if $\varphi \sim \psi$, then $\omega_\varphi^r(f, t)_{w,p} \sim \omega_\psi^r(f, t)_{w,p}$.

In the proofs of Theorems 1–3, we shall need the concept of (weighted) main-part moduli. To avoid unnecessary complications, let us consider only the case $(a, b) = (0, \infty)$ as $(a, b) = (0, 1)$ can be handled similarly (see [4, Sect. 6.2 and Appendix B]) and as for $(a, b) = (-\infty, \infty)$, the main-part moduli and the moduli defined above coincide.

If $\beta(0) \geqslant 1$, we write

$$\Omega_\varphi^r(f, t)_{w,p} = \omega_\varphi^r(f, t)_{w,p} = \sup_{0 < h \leqslant t} \|w\Delta_{h\varphi}^r f\|_p.$$

In the remaining case, $0 \leqslant \beta(0) < 1$, we write

$$\Omega_\varphi^r(f, t)_{w,p} = \sup_{0 < h \leqslant t} \|w\Delta_{h\varphi}^r f\|_{L_p(h^*, \infty)}, \tag{5.3}$$

where $h^* = (rh)^{1/(1-\beta(0))}$ (see (2.5)). Thus, Ω and w differ only for $\beta(0) < 1$ and the difference in their definition is that in the expression of Ω, we avoid a small neighbourhood of 0. Let us mention that the cases $\beta(0) \geqslant 1$ and $\beta(0) < 1$ are different because $\beta(0) \geqslant 1$ implies

$$(x - rh\varphi(x)/2, x + rh\varphi(x)/2) \subseteq (0, \infty)$$

for small h and every $x > 0$, while for $0 \leqslant \beta(0) < 1$, this is true only if

$$x \geqslant (1/2)^{1/(1-\beta)} (rh)^{1/(1-\beta)} = (1/2)^{1/(1-\beta)} h^*.$$

The corresponding main-part K-functional is given by

$$\mathscr{K}_{r,\varphi}(f, t^r)_{w,p} \equiv K_{r,\varphi}(f, t^r)_{w,p}$$

if $\beta(0) \geqslant 1$ and

$$\mathscr{K}_{r,\varphi}(f, t^r)_{w,p} \equiv \sup_{0 < h \leqslant t} \inf_{g^{(r-1)} \in AC_{\text{loc}}} \{\|w(f-g)\|_{L_p(h^*, \infty)} + h^r \|w\varphi^r g^{(r)}\|_{L_p(h^*, \infty)}\}$$

when $0 \leqslant \beta(0) < 1$.

In [4, Theorem 6.2.1], we proved

$$\mathscr{K}_{r,\varphi}(f, t^r)_{w,p} \sim \Omega_\varphi^r(f, t)_{w,p} \qquad (t \to 0). \tag{5.4}$$

Using this equivalence, we derived in [4, Sects. 6.2–6.4] many properties of Ω which display complete analogy with ordinary moduli of smoothness, for example (see [4, Theorem 6.2.5]),

$$\Omega_\varphi^{r+1}(f, t)_{w,p} \ll \Omega_\varphi^r(f, t)_{w,p}, \tag{5.5}$$

and for $\lambda \geqslant 1$ (see [4, (6.2.8)]),

$$\Omega_\varphi^r(f, \lambda t)_{w,p} \leqslant \lambda^r \Omega_\varphi^r(f, t)_{w,p}.$$

The converse of (5.5) is the Marchaud-type inequality [4, Theorem 6.4.2],

$$\Omega_\varphi^r(f, t)_{w,p} \leqslant t^r \left(\int_t^1 (\Omega_\varphi^{r+1}(f, u)_{w,p}/u^{r+1})\, du + \|wf\|_p \right). \tag{5.6}$$

In [4], we considered more general weights than the ones above and (5.6) was proved under the assumption that

$$\gamma(c) + r\beta(c) > -1/p$$

if c ($c = a$ or $c = b$) is a finite endpoint with $0 \leqslant \beta(c) < \infty$, but as $\gamma(c) \geqslant 0$, this is always true in the case dealt with here.

Finally, the moduli ω and Ω are connected by [4, Theorem 6.2.2]

$$\Omega_\varphi^r(f, t)_{w,p} \leqslant \omega_\varphi^r(f, t)_{w,p} \leqslant \int_0^t (\Omega_\varphi^r(f, u)_{w,p}/u)\, du. \tag{5.7}$$

In particular, for $\alpha > 0$,

$$\omega_\varphi^r(f, t)_{w,p} = \mathcal{O}(t^\alpha) \Leftrightarrow \Omega_\varphi^r(f, t)_{w,p} = \mathcal{O}(t^\alpha). \tag{5.8}$$

In the next section, the following lemmas will be used extensively.

LEMMA 1. *Suppose W is a weight function on an interval (c, d), $0 \leqslant c \leqslant 2c \leqslant d$, and $W(x) \sim W(y)$ if $x \sim y$. Then for $0 \leqslant i \leqslant r$,*

$$\|W(x)\, x^i g^{(i)}(x)\|_{L_p(c,d)} \leqslant \|W(x)\, g(x)\|_{L_p(c,d)} + \|W(x)\, x^r g^{(r)}(x)\|_{L_p(c,d)}.$$

Proof. We make use of the inequality [3, p. 310]

$$\|g^{(i)}\|_{L_p(I)} \leqslant |I|^{-i} \|g\|_{L_p(I)} + |I|^{r-i} \|g^{(r)}\|_{L_p(I)} \tag{5.9}$$

where I denotes an arbitrary interval. We can decompose (c, d) into disjoint intervals $I_k = (\xi_k, \eta_k)$ with $1 \leqslant (\eta_k - \xi_k)/\xi_k \leqslant 4$ for every k. Then (5.9) implies

$$\begin{aligned} \|W(x)\, x^i g^{(i)}(x)\|_{L_p(I_k)}^p &\leqslant W^p(\xi_k)\, |I_k|^{ip}\, \|g^{(i)}\|_{L_p(I_k)}^p \\ &\leqslant W^p(\xi_k)\, \|g\|_{L_p(I_k)}^p + W^p(\xi_k)\, |I_k|^{rp}\, \|g^{(r)}\|_{L_p(I_k)}^p \\ &\leqslant \|Wg\|_{L_p(I_k)}^p + \|W(x)\, x^r g^{(r)}(x)\|_{L_p(I_k)}^p. \end{aligned}$$

Summing these up for all k and taking the pth root, we obtain the statement of the lemma. ∎

LEMMA 2. *If W is a weight function on an interval (c, d), $d \geqslant c+1$, and $W(x) \sim W(y)$ when $x-y \sim 1$, then for $0 \leqslant i \leqslant r$,*

$$\|Wg^{(i)}\|_{L_p(c,d)} \ll \|Wg\|_{L_p(c,d)} + \|Wg^{(r)}\|_{L_p(c,d)}.$$

Proof. The proof coincides with the preceding one if we divide (c, d) into intervals $\{I_k\}$ of length between 1 and 2. ∎

LEMMA 3. *Suppose $0 \leqslant c \leqslant 1$ and $g(x)=0$ for $x \geqslant 1$. Then, under the assumptions*

$$\rho > -1/p, \tag{5.10}$$

and $g^{(r-1)} \in AC_{\mathrm{loc}}$, we have for $1 \leqslant i \leqslant r$,

$$\|x^\rho g^{(i)}\|_{L_p(c,1)} \ll \|x^{\rho+r-i} g^{(r)}\|_{L_p(c,1)} \tag{5.11}$$

uniformly in c.

Proof. It is enough to prove the statement for $i=0$ and $r=1$, since the general case follows from it by iteration.

Our assumptions imply that

$$g(x) = -\int_x^\infty g'(u)\, du.$$

Applying Hardy's inequality (see [5, p. 273])—and this is where we need (5.10)—to the function

$$g^*(x) = -\int_x^\infty g'(u)\, \chi(c, 1)\, du,$$

where $\chi(c, 1)$ is the characteristic function of the interval $(c, 1)$, we have

$$\|x^\rho g\|_{L_p(c,1)} = \|x^\rho g^*\|_{L_p(c,1)} \leqslant \frac{p}{p\rho+1} \|x^{\rho+1}(g^*)'\|_{L_p(0,\infty)}$$

$$= \frac{p}{p\rho+1} \|x^{\rho+1} g'\|_{L_p(c,1)}. \quad ∎$$

LEMMA 4. *If*

$$\int_0^1 (\Omega_\varphi^r(f, t)_{w,p}/t^{s+1})\, dt < \infty,$$

then $f^{(s-1)} \in AC_{\text{loc}}$,

$$\Omega_\varphi^{r-s}(f^{(s)}, t)_{w\varphi^s, p} \ll \int_0^t (\Omega_\varphi^r(f, u)_{w,p}/u^{s+1})\, du \tag{5.12}$$

and

$$\|w\varphi^s f^{(s)}\|_p \ll \int_0^1 (\Omega_\varphi^r(f, t)_{w,p}/t^{s+1})\, dt + \|wf\|_p. \tag{5.13}$$

Proof. The first assertion of the lemma and (5.12) is the content of [4, Theorem 6.3.1]. $f^{(s-1)} \in AC_{\text{loc}}$ now implies

$$\lim_{t\to 0} \Delta_{t\varphi(x)}^s f(x)/t^s = \varphi^s(x)\, f^{(s)}(x) \qquad \text{(a.e.)}$$

and, therefore, using Fatou's lemma we have

$$\|w\varphi^s f^{(s)}\|_p \leqslant \liminf_{t\to 0+} \Omega_\varphi^s(f, t)_{w,p}/t^s. \tag{5.14}$$

According to (5.6),

$$\Omega_\varphi^s(f, t)_{w,p}/t^s \ll \int_t^1 (\Omega_\varphi^{s+1}(f, u)_{w,p}/u^{s+1})\, du + \|wf\|_p.$$

The iteration of this yields

$$\begin{aligned} \Omega_\varphi^s(f, t)_{w,p}/t^s &\ll \int_t^1 \int_{t_1}^1 \cdots \int_{t_{r-s-1}}^1 (\Omega_\varphi^r(f, u)_{w,p}/u^r)\, du\, dt_{r-s-1} \cdots dt_1 + \|wf\|_p \\ &\ll \int_t^1 (\Omega_\varphi^r(f, u)_{w,p}/u^{s+1})\, du + \|wf\|_p \end{aligned}$$

which, together with (5.14), proves (5.13). ∎

6. Proofs of Theorems 1–3

Proof of Theorem 1. We carry out the proof first for the case $(a, b) = (0, \infty)$, because in this case we must deal with the difficulties caused by both finite and infinite endpoints. The other cases, that is, $(a, b) = (0, 1)$ and $(a, b) = (-\infty, \infty)$, will be handled similarly at the end of the proof.

Thus, let $(a, b) = (0, \infty)$. We may assume that

$$\varphi(x) = x^{\beta(0)} \qquad \text{on } (0, 1)$$

and

$$\varphi(x) = x^{\beta(\infty)} \qquad \text{on } (2, \infty).$$

Let $\psi_1, \psi_2 \in C^r(0, \infty)$ be monotone functions such that

$$\psi_1(x) = \begin{cases} 1 & \text{if } x < 1/2 \\ 0 & \text{if } x > 3/4 \end{cases}$$

and

$$\psi_2(x) = \begin{cases} 0 & \text{if } x < 3 \\ 1 & \text{if } x > 4. \end{cases}$$

We write

$$f = f\psi_1 + f(1 - \psi_1)(1 - \psi_2) + f\psi_2 \equiv f_1 + f_2 + f_3.$$

It follows easily from Theorems A and B, given in Section 4, and from (5.9) that

$$\omega_\varphi^r(f_i, t)_{w,p} \ll \omega_\varphi^r(f, t)_{w,p} + t^r \|wf\|_p,$$

and for $F_i = f_i \circ \theta$,

$$\omega_1^r(F_i, t)_{(w \circ \theta)(\varphi \circ \theta)^{1/p}, p} \leqslant \omega_1^r(F, t)_{(w \circ \theta)(\varphi \circ \theta)^{1/p}, p} + \|(w \circ \theta)(\varphi \circ \theta)^{1/p} F\|_p.$$

Therefore, it is sufficient to prove Theorem 1 separately for f_1, f_2, and f_3. It will be easiest to prove (3.3) and (3.4) of Theorem 1 for f_2 because the support of f_2 lies in $[1/2, 4]$. In fact, by the proof of Theorem A or by Theorem A and (5.9), there exists, for every $0 < t < 1$, a function g such that $\operatorname{supp} g \subseteq (1/4, 5)$ and

$$\|w(f_2 - g)\|_p + t^r \|w\varphi^r g^{(r)}\|_g \ll \omega_\varphi^r(f_2, t)_{w,p}. \tag{6.1}$$

Making use of the substitution $x = \theta(u)$ and the formula $\theta' = \varphi \circ \theta$ (cf. (3.1)), we have

$$\|(w \circ \theta)(\varphi \circ \theta)^{1/p} (f_2 \circ \theta - g \circ \theta)\|_p \ll \omega_\varphi^r(f_2, t)_{w,p}. \tag{6.2}$$

As the functions f_2 and g vanish outside $(1/4, 5)$, we will be interested in the interval $(\theta^{-1}(1/4), \theta^{-1}(5))$. On $(\theta^{-1}(1/4), \theta^{-1}(5))$, we have

$$w \circ \theta \leqslant C, \qquad |\theta^{(i)}| \leqslant C, \quad 1 \leqslant i \leqslant r,$$

with some constant C. Furthermore, $(g \circ \theta)^{(r)}$ is the linear combination of terms of the form $(g^{(i)} \circ \theta)\, \theta^{(i_1)} \cdots \theta^{(i_s)}$ which satisfy

$$\|(g^{(i)} \circ \theta)\, \theta^{(i_1)} \cdots \theta^{(i_s)}\|_p \leqslant \|g^{(i)} \circ \theta\|_p.$$

Lemma 2 now implies

$$\begin{aligned} &\|(w \circ \theta)(\varphi \circ \theta)^{1/p}\, (g \circ \theta)^{(r)}\|_p \\ &\quad \leqslant \|(g \circ \theta)^r\|_p \leqslant \sum_{i=1}^{r} \|g^{(i)} \circ \theta\|_p \\ &\quad \leqslant \sum_{i=1}^{r} \|g^{(i)}\|_p \leqslant \|g^{(r)}\|_p + \|g\|_p \\ &\quad \leqslant \|w\varphi^r g^{(r)}\|_p + \|w(f_2 - g)\|_p + \|wf_2\|. \end{aligned} \tag{6.3}$$

The last step of (6.3) follows from the fact that w and φ are bounded away from zero on $(1/4, 5)$ and hence, on the support of g and f_2. The relations (6.2) and (6.3) imply via Theorem A and (6.1) that (3.3) is valid for f_2. The inequality (3.4) for f_2 can be demonstrated similarly.

The proof of (3.3) and (3.4) for f_1 and f_3 requires more sophisticated arguments. Let us begin with f_1.

Proof of Theorem 1 *for* f_1. We write $f = f_1$ and thus suppose that $f(x) = 0$ for $x \geqslant 3/4$. First we will show that for every $t > 0$, there is a function $g = g_t$ such that g vanishes on $(4/5, \infty)$ and

$$\|w(f - g)\|_p + t^r\, \|w\varphi^r g^{(r)}\|_p \leqslant \omega_\varphi^r(f, t)_{w,p}. \tag{6.4}$$

In fact without the requirement that g vanishes on $(4/5, \infty)$, (6.4) follows from Theorem A of Section 5 and from the definition of the K-functional $K_{r,\varphi}(f, t^r)_{w,p}$. But if (6.4) is valid for some g (not necessarily supported on $[0, 4/5]$) which we call g_*, we can show that it is also valid with $g = g_* \psi$ where $\psi \in C^\infty(0, \infty)$, $\psi(x) = 1$ for $x \in [0, 3/4]$, and $\psi(x) = 0$ for $x \in [4/5, \infty)$ (4/5 rather than 1 is chosen to suit the case (0, 1) to be proven later as well as the present case). While the estimate of $\|w(f - g)\|$ by $\|w(f - g_*)\|$ is easy, we have to use (5.9) to obtain the estimate of $\|w\varphi^r g^{(r)}\|_{L_p[3/4, 4/5]}$ by $\|w\varphi^r g_*^{(r)}\|_{L_p[3/4, 4/5]}$ and $\|wg_*\|_{L_p[3/4, 4/5]}$. (Clearly $\|w\varphi^r g^{(r)}\|_{L_p[0, 3/4]} = \|w\varphi^r g_*^{(r)}\|_{L_p[0, 3/4]}$ and $\|w\varphi^r g^{(r)}\|_{L_p[4/5, \infty)} = 0$.)

The estimate (6.4) immediately implies

$$\|(w \circ \theta)(\theta')^{1/p}\, (f \circ \theta - g \circ \theta)\|_p \leqslant \omega_\varphi^r(f, t)_{w,p}. \tag{6.5}$$

If we can show that

$$\|(w \circ \theta)(\theta')^{1/p}\, (g \circ \theta)^{(r)}\|_p \leqslant \|w\varphi^r g^{(r)}\|_p + \|wg\|_p, \tag{6.6}$$

the estimate (3.3) will follow from (6.4)-(6.6), Theorems A and B, and $\theta' = \varphi \circ \theta$.

Also, for every $t > 0$, there exists $G = G_t$ such that G vanishes on $\theta^{-1}(1, \infty)$ and

$$\|(w \circ \theta)(\varphi \circ \theta)^{1/p}(f \circ \theta - G)\|_p + t^r \|(w \circ \theta)(\varphi \circ \theta)^{1/p} G^{(r)}\|_p$$
$$\leqslant \omega_1^r(f \circ \theta, t)_{(w \circ \theta)(\varphi \circ \theta)^{1/p}, p}.$$

To prove (3.4), which is the converse of (3.3), it is sufficient to show that

$$\|w\varphi^r(G \circ \theta^{-1})^{(r)}\|_p \leqslant \|(w \circ \theta)(\varphi \circ \theta)^{1/p} G^{(r)}\|_p$$
$$+ \|(w \circ \theta)(\varphi \circ \theta)^{1/p} G\|_p. \tag{6.7}$$

Thus, it remains to prove only (6.6) and (6.7). We will prove (6.6) and (6.7) for the cases $0 \leqslant \beta(0) < 1$, $\beta(0) = 1$, and $\beta(0) > 1$ separately. Note that we assume $\varphi(x) = x^{\beta(0)}$ on $(0, 1)$.

Case I: $0 \leqslant \beta(0) < 1$. We set $\beta = \beta(0)$ and hence, $\varphi(x) = x^\beta$ on $[0, 1]$. The endpoint A (where θ maps (A, B) onto $(0, \infty)$, cf. Section 3) is finite, and by linear substitution (see Remark 3 in Section 3), we can assume that

$$\Gamma(x) = x^{1-\beta}, \qquad \theta(x) = x^{1/(1-\beta)} \equiv x^\delta, \qquad \text{and} \qquad (A, B) = (0, \infty).$$

Therefore, the expression

$$((g \circ \theta)(u))^{(r)} = (g(u^\delta))^{(r)}$$

is the linear combination of terms of the type

$$g^{(i)}(u^\delta)\, u^{i\delta - r}, \qquad 1 \leqslant i \leqslant r.$$

Hence, to prove (6.6), we have to estimate

$$\|(w \circ \theta)(u)(\theta'(u))^{1/p} g^{(i)}(\theta(u))\, u^{i\delta - r}\|_p$$

for $1 \leqslant i \leqslant r$, which, making use of the change of variable $x = \theta(u)$, $u = x^{1-\beta}$ becomes

$$I \equiv \|w(x)\, g^{(i)}(x)\, x^{i - r(1-\beta)}\|_p. \tag{6.8}$$

Since $w(x) \sim x^{\gamma(0)}$ on $(0, 1)$, Lemma 3 and the condition

$$\gamma(0) + i - r(1 - \beta(0)) > -\frac{1}{p}, \tag{6.9}$$

which is valid if we assume (3.2), yield

$$\|w(x)\, g^{(i)}(x)\, x^{i-r(1-\beta)}\|_p \leqslant \|w(x)\, x^{r\beta} g^{(r)}(x)\|_p,$$

and this proves (6.6).

We will prove (6.7) in a similar way. The expression

$$(G \circ \theta^{-1}(x))^{(r)} = (G(x^{1-\beta}))^{(r)}$$

consists of terms such as

$$G^{(i)}(x^{1-\beta})\, x^{-i\beta-(r-i)}, \qquad 1 \leqslant i \leqslant r,$$

and hence, we have to estimate for $1 \leqslant i \leqslant r$, the expression

$$\begin{aligned} J &= \|w(x)\, x^{r\beta} G^{(i)}(x^{1-\beta})\, x^{-i\beta-(r-i)}\|_p \\ &= \|(w \circ \theta)(u)(\theta'(u))^{1/p}\, G^{(i)}(u)\, u^{i-r}\|_p. \end{aligned} \tag{6.10}$$

Since

$$(w \circ \theta)(u)(\theta'(u))^{1/p} \sim u^{\gamma(0)/(1-\beta)+\beta/(1-\beta)p},$$

Lemma 3 implies that if

$$\frac{\gamma(0)}{1-\beta(0)} + \frac{\beta(0)}{(1-\beta(0))\,p} + i - r > -\frac{1}{p} \qquad \text{for} \quad 1 \leqslant i \leqslant r, \tag{6.11}$$

then

$$J \leqslant \|(w \circ \theta)(\theta')^{1/p}\, G^{(r)}\|_p,$$

i.e., (6.7) is valid. The assumption (6.11) for $i=1$ is actually stronger than (6.11) for $i>1$ and for $i=1$, it is exactly (3.2).

Thus the proof of (6.6) and (6.7) for f_1 is complete in the case $0 \leqslant \beta(0) < 1$.

Case II: $\beta(0)=1$. In this case, we can choose

$$\Gamma(x) = \int_1^x \frac{du}{u} = \log x, \qquad \theta(x) = e^x$$

and we have $(A, B) = (-\infty, \infty)$. The expression

$$(g(\theta(u)))^{(r)} = (g(e^u))^{(r)}$$

consists of terms like

$$g^{(l)}(e^u)\, e^{lu} \qquad \text{for} \quad 1 \leqslant l \leqslant r.$$

Hence, we have to show

$$\|(w\circ\theta)(\theta')^{1/p}g^{(l)}(\theta)\,\theta^l\|_{L_p(-\infty,\infty)}=\|w(x)\,x^l g^{(l)}(x)\|_{L_p(0,\infty)}$$
$$\ll \|w(x)\,x^r g^{(r)}(x)\|_p+\|wg\|_p$$

which follows from Lemma 1 of Section 5. This proves (6.6). For the proof of (6.7), we observe that

$$(G(\log x))^{(r)}$$

consists of terms such as

$$G^{(l)}(\log x)\,x^{-r},\qquad 1\leqslant l\leqslant r.$$

We can now apply Lemma 2 of Section 5 to obtain

$$\|w(x)\,x^r G^{(l)}(\log x)\,x^{-r}\|_p=\|w(e^u)\,e^{u/p}G^{(l)}(u)\|_p$$
$$\ll \|w(e^u)\,e^{u/p}G^{(r)}(u)\|_p+\|w(e^u)\,e^{u/p}G(u)\|_p$$

which implies (6.7).

Case III: $\beta=\beta(0)>1$. For $\beta>1$, we have $(A,B)=(-\infty,\infty)$ and we can choose

$$\Gamma(x)=-x^{1-\beta},\qquad \theta(x)=(-x)^{1/(1-\beta)}.$$

In this case, the consideration is very similar to that of Case I above. There are only two differences. The first one is that because of

$$-r(1-\beta)>0,$$

we can apply Lemma 1 instead of Lemma 3 and so we need no extra assumption in the proof of (6.6). The second one is that the norm in J, given by (6.10), is actually a norm on $(-\infty,-1)$ (note that $\operatorname{supp} G\subseteq \theta^{-1}((0,1))\subseteq(-\infty,-1)$), hence, we can apply Lemma 1 (more precisely its $(-\infty,0)$-variant) to conclude

$$J\ll \|(w\circ\theta)(\theta')^{1/p}G^{(r)}\|_p+\|(w\circ\theta)(\theta')^{1/p}G\|_p$$

and here, again, we do not need any assumption on the parameters $\gamma(0)$, r, and $\beta(0)>1$.

Cases I–III prove Theorem 1 for $f=f_1$.

Finally, we verify Theorem 1 for f_3.

Proof of Theorem 1 *for* f_3. Let $f=f_3$. Then $\operatorname{supp} f\subseteq(3,\infty)$ and $\varphi(x)=x^\beta$, $\beta=\beta(\infty)\leqslant 1$ on $(2,\infty)$.

We follow the consideration used in the proof for f_1. The functions g and G in this case can be chosen in such a way that

$$\operatorname{supp} g \subseteq (2, \infty) \qquad \text{and} \qquad \operatorname{supp} G \subseteq \theta^{-1}((2, \infty)).$$

We have to prove again (6.6) and (6.7). We distinguish two cases.

Case IV: $\beta = \beta(\infty) = 1$. We can choose

$$\Gamma(x) = \int_1^x \frac{du}{u} = \log x, \qquad \theta(x) = e^x$$

and the proof coincides with that of Case II above.

Case V: $\beta = \beta(\infty) < 1$. We can choose

$$\Gamma(x) = x^{1-\beta}, \qquad \theta(x) = x^{1/(1-\beta)}$$

and follow the proof of Cases I and III above. Since in this case, both in (6.8) and in (6.10), the support of the function is contained in $(1, \infty)$, we can apply Lemma 1 of Section 5 in estimating I from (6.8) and J from (6.10) to obtain

$$I \ll \|w\varphi^r g^{(r)}\|_p + \|wg\|_p$$

and

$$J \ll \|(w \circ \theta)(\theta')^{1/p} G^{(r)}\|_p + \|(w \circ \theta)(\theta')^{1/p} G\|_p$$

and this completes the proof, when $(a, b) = (0, \infty)$.

Let now $(a, b) = (0, 1)$. We use the function ψ_1 from the preceding part of the proof and set

$$\psi_3(x) \equiv \psi_1(1-x) = \begin{cases} 1 & \text{if } x > 1/2 \\ 0 & \text{if } x < 1/4. \end{cases}$$

We can now write

$$f = f\psi_1 + f(1-\psi_1)(1-\psi_3) + f\psi_3 \equiv f_1 + f_2 + f_3.$$

The estimates for f_2 and f_1 are the same as in the proof for the case $(0, \infty)$. (Note that f_2 has support in $[1/4, 3/4]$ and f_1 has support in $[0, 3/4]$.)

Finally, the estimate of f_3 is parallel to that of f_1. (Note that the support of f_3 is contained in $[1/4, 1]$.) To show this we just replace x by $(1-x)$. An alternative way is to use the substitution $u = 1-x$, $\bar{w}(u) = w(x)$, $\bar{f}(u) = f(x)$. Then in the decomposition

$$\bar{f} = \bar{f}\psi_1 + \bar{f}(1-\psi_1)(1-\psi_3) + \bar{f}\psi_3 \equiv \bar{f}_1 + \bar{f}_2 + \bar{f}_3,$$

$\bar{f}_1(u)$ coincides with $f_3(x)$, and so the proof applied to $\bar{f}$ and $\bar{f}_1$ rather than to f and f_1 yields the necessary estimates for f_3.

Finally, the case $(a, b) = (-\infty, \infty)$ is similar if we use the decomposition

$$f = f\psi_1 + f(1-\psi_1)(1-\psi_2) + f\psi_2 \equiv f_1 + f_2 + f_3,$$

where $\operatorname{supp} f_1 \subseteq (-\infty, 3/4)$, $\operatorname{supp} f_2 \subseteq [1/2, 4]$, $\operatorname{supp} f_3 \subseteq [3, \infty)$ and in this case the proof for f_1 and f_3 will be parallel to the proof for f_3 in the case $(a, b) = (0, \infty)$. The proof for f_2 is exactly the same as in the case $(a, b) = (0, \infty)$. ▮

Proof of Theorem 2. Assuming (3.6), we know that $f^{(s-1)}$ is locally absolutely continuous and

$$\Omega_\varphi^{r-s}(f^{(s)}, t)_{w\varphi^s, p} \leqslant \int_0^t (\omega_\varphi^r(f, u)_{w,p}/u^{s+1})\, du, \tag{6.12}$$

from Lemma 4. By slightly modifying the proof of Theorem 1, we can show that it holds for the main part moduli as well. (Here (5.4) should be used instead of Theorems A and B.) Therefore, assuming (3.5), we have

$$\Omega_1^{r-s}(f^{(s)} \circ \theta, t)_{(w\circ\theta)(\varphi\circ\theta)^{s+1/p}, p} \leqslant \Omega_\varphi^{r-s}(f^{(s)}, t)_{w\varphi^s, p} + t^{r-s}\, \|w\varphi^s f^{(s)}\|_p.$$

(See also [4, Sect. 6.2] about the freedom we have in choosing h^* in (2.5).) By [4, Theorem 6.3.1(b)], we have

$$\Omega_1^r(F, t)_{(w\circ\theta)(\varphi\circ\theta)^{s+1/p}, p} \leqslant t^s \Omega_1^{r-s}(F^{(s)}, t)_{(w\circ\theta)(\varphi\circ\theta)^{s+1/p}, p}. \tag{6.13}$$

Furthermore, (5.7) yields

$$\omega_1^r(F, t)_{W,p} \leqslant \int_0^t (\Omega_1^r(F, u)_{W,p}/u)\, du.$$

Combining the above, we have

$$\begin{aligned} \omega_1^r(F, t)_{(w\circ\theta)(\varphi\circ\theta)^{s+1/p}, p} &\leqslant \int_0^t u^{s-1} \int_0^u (\omega_\varphi^r(f, v)_{w,p}/v^{s+1})\, dv\, du \\ &\quad + \int_0^t u^{r-1}\, \|w\varphi^s f^{(s)}\|_p\, du \\ &\leqslant t^s \int_0^t (\omega_\varphi^r(f, v)_{w,p}/v^{s+1})\, dv + t^r\, \|w\varphi^s f^{(s)}\|_p. \end{aligned}$$

The relation (3.7) now follows from (5.13) and the above. We omit the proof of (3.8) as it is very similar. ▮

Proof of Theorem 3. According to (5.5) and (5.6),

$$\Omega_\varphi^{r+1}(f, t)_{w,p} \ll \Omega_\varphi^r(f, t)_{w,p}$$

and

$$\Omega_\varphi^r(f, t)_{w,p} \ll t^r \left(\int_t^1 (\Omega_\varphi^{r+1}(f, u)_{w,p}/u^{r+1})\, du + \|wf\|_p \right).$$

Hence, $wf \in L_p$ and

$$\Omega_\varphi^r(f, t)_{w,p} = \mathcal{O}(t^\alpha)$$

imply

$$\Omega_\varphi^{s+2}(f, t)_{w,p} = \mathcal{O}(t^\alpha).$$

Making use of Lemma 4 (of Section 5), we can derive

$$\Omega_\varphi^2(f^{(s)}, t)_{w\varphi^s, p} = \mathcal{O}(t^{\alpha-s}), \tag{6.14}$$

where s satisfies $0 < \alpha - s \leqslant 1$. Thus, if we can show for $0 < \rho \leqslant 1$ that

$$\Omega_\varphi^2(f, t)_{w,p} = \mathcal{O}(t^\rho) \tag{6.15}$$

is equivalent to

$$\Omega_1^2(f \circ \theta, t)_{(w \circ \theta)(\varphi \circ \theta)^{1/p}, p} = \mathcal{O}(t^\rho) \tag{6.16}$$

whenever $wf \in L_p$, we will apply this equivalence to the function $f^{(s)}$ (rather than f) and the weight function $w\varphi^s$ (rather than w) with $\rho = \alpha - s$ to obtain

$$\Omega_1^2(F^{(s)}, t)_{(w \circ \theta)(\varphi \circ \theta)^{s+1/p}, p} = \mathcal{O}(t^\rho).$$

We now use (6.14), (5.8), and (6.13) to derive (3.10). The verification that (3.10) implies (3.9) is identical, we just have to reverse our steps.

Thus, it remains to prove that (6.15) and (6.16) are equivalent. For $\beta(c) \geqslant 1$ when c ($c = a$ or $c = b$) is a finite endpoint, $\Omega \equiv \omega$ and the equivalence of (6.15) and (6.16) follows from Theorem 1.

When $0 \leqslant \beta(c) < 1$ we can no longer apply Theorem 1 to verify that (6.15) and (6.16) are equivalent, since in Theorem 3 we do not have the assumption (3.2) of Theorem 1. In this case more delicate consideration is needed.

Suppose that at some finite endpoint c we have $0 \leqslant \beta(c) < 1$. For simplicity, we set $(a, b) = (0, \infty)$ and $0 \leqslant \beta = \beta(0) < 1$. (The case $(a, b) = (0, 1)$ with $\beta(0) < 1$ or $\beta(1) < 1$ can be treated similarly.)

We now follow the proof of Case I of Theorem 1.

Thus, we may suppose that $f(x) = 0$ on $(3/4, \infty)$, $\varphi(x) = x^{\beta}$, $0 \leqslant \beta = \beta(0) < 1$ on $(0, 1)$, $\Gamma(x) = x^{1-\beta}$, and $\theta(x) = x^{1/(1-\beta)}$. We can choose functions g and G such that $\operatorname{supp} g \subseteq (0, 1)$, $\operatorname{supp} G \subseteq (0, 1)$,

$$\|w(f-g)\|_{L_p(t^*, \infty)} + t^2 \|w\varphi^2 g''\|_{L_p(t^*, \infty)} \leqslant \Omega_\varphi^2(f, t)_{w,p}$$

(with $t^* = (2t)^{1/(1-\beta)}$) and

$$\begin{aligned}\|(w\circ\theta)(\varphi\circ\theta)^{1/p}(f\circ\theta - G)\|_{L_p(2t,\infty)} &+ t^2 \|(w\circ\theta)(\varphi\circ\theta)^{1/p} G''\|_{L_p(2t,\infty)} \\ &\leqslant \Omega_1^2(f\circ\theta, t)_{(w\circ\theta)(\varphi\circ\theta)^{1/p}, p}.\end{aligned}$$

Following the proof of Theorem 1, it is enough to show that for $\rho \leqslant 1$,

$$\|w\varphi^2 g''\|_{L_p(t^*, \infty)} \leqslant t^{\rho-2} \tag{6.17}$$

implies

$$\|w(x)\,\varphi^2(x)\,g'(x)/x\|_{L_p(t^*, \infty)} \leqslant t^{\rho-2} \tag{6.18}$$

and

$$\|(w\circ\theta)(\varphi\circ\theta)^{1/p} G''\|_{L_p(2t,\infty)} \leqslant t^{\rho-2} \tag{6.19}$$

implies

$$\|(w\circ\theta(u))(\varphi\circ\theta(u))^{1/p} G'(u)/u\|_{L_p(2t,\infty)} \leqslant t^{\rho-2}. \tag{6.20}$$

To prove these implications, we need the following lemma.

Lemma 5. *Suppose*

$$\gamma - 1 + 1/p + (2-\rho)/\delta > 0, \qquad \delta > 0, \quad \rho < 2. \tag{6.21}$$

Then

$$\left\{\int_{(2t)^\delta} |x^\gamma g''(x)|^p\,dx\right\}^{1/p} \leqslant t^{\rho-2} \qquad (t \to 0+) \tag{6.22}$$

implies for $g, g' \in AC_{\text{loc}}$ *with* $\operatorname{supp} g \subseteq (0, 1)$, *that*

$$\left\{\int_{(2t)^\delta} |x^{\gamma-1} g'(x)|^p\,dx\right\}^{1/p} \leqslant t^{\rho-2} \qquad (t \to 0+).$$

Assuming Lemma 5, the implication (6.17) $\Rightarrow$ (6.18) follows from

$$w(x)\,\varphi^2(x) \sim x^{\gamma(0)+2\beta},$$

where $\gamma(0) \geqslant 0$, $t^* = (2t)^{1/(1-\beta)}$, and

$$(\gamma(0)+2\beta)-1+1/p+(2-\rho)/(1/(1-\beta)) \geqslant 2\beta-1+1/p+(1-\beta)>0$$

because $\rho \leqslant 1$.

Similarly, the implication (6.19) $\Rightarrow$ (6.20) follows from Lemma 5 since

$$(w \circ \theta(u))(\varphi \circ \theta(u))^{1/p} \sim u^{(\gamma(0)+\beta/p)/(1-\beta)}$$

and for $\rho \leqslant 1$, $\gamma(0) \geqslant 0$,

$$(\gamma(0)+\beta/p)/(1-\beta)-1+1/p+(2-\rho)/1>0. \quad ▮$$

Proof of Lemma 5. The relation (6.22) implies that

$$\left\{\int_t |x^\gamma g''(x)|^p\,dx\right\}^{1/p} \ll t^{(\rho-2)/\delta}$$

and therefore, using Hölder's inequality, we have

$$\begin{aligned}\int_t^{2t} |g''(x)|\,dx &\leqslant t^{1-1/p}\left\{\int_t^{2t} |g''(x)|^p\,dx\right\}^{1/p}\\ &\ll t^{-\gamma+1-1/p}\left\{\int_t^{2t} |x^\gamma g''(x)|^p\,dx\right\}^{1/p}\\ &\ll t^{-\gamma+1-1/p+(\rho-2)/\delta}.\end{aligned}$$

Adding these together for $t, 2t, 4t, \ldots$ and utilizing the fact that the exponent $-\gamma+1-1/p+(\rho-2)/\delta$ is negative (which follows from (6.21)), we obtain

$$|g'(t)| \leqslant \int_t^1 |g''(x)|\,dx \ll t^{-\gamma+1-1/p+(\rho-2)/\delta}.$$

Hence,

$$\left\{\int_{(2t)^\delta}^1 |x^{\gamma-1} g'(x)|^p\,dx\right\}^{1/p} \ll \left\{\int_{(2t)^\delta}^1 x^{(\rho-2)p/\delta-1}\,dx\right\}^{1/p} \ll t^{\rho-2}$$

which completes the proof. ▮

7. Applications

In this section, we apply our results to some problems in approximation theory.

Let $\alpha > 0$, and s be an integer such that $s < \alpha \leqslant s+1$. In the course of the proof of Theorem 3, we verified the following proposition.

Proposition. *For* $0 < \alpha < r$,

$$\Omega_\varphi^r(f, t)_{w,p} = \mathcal{O}(t^\alpha) \Leftrightarrow \Omega_1^2(f^{(s)} \circ \theta, t)_{(w \circ \theta)(\varphi \circ \theta)^{s+1/p}, p} = \mathcal{O}(t^{\alpha - s}). \tag{7.1}$$

We shall use this equivalence relation in our first application. Let us mention that the proof of (7.1) given above actually works for $p = \infty$ as well (unlike the proofs of Theorems 1 and 2) if $\beta(c) > 0$ when c ($c = a$ or $c = b$) is a finite endpoint.

Note furthermore, that we have a certain freedom in choosing θ (see Remarks 2 and 3 in Section 3).

1. Suppose $(a, b) = (-1, 1)$, $w(x) = (1+x)^{\gamma_1}(1-x)^{\gamma_2}$ with $\gamma_i \geqslant 0$ (w is a Jacobi weight), and

$$E_n(f)_{w,p} = \inf_{\deg P_n \leqslant n} \|w(f - P_n)\|_{L_p(-1,1)}$$

is the best approximation of f by (algebraic) polynomials of degree at most n. It was shown in [4, Corollary 8.2.2] that for $0 < \alpha < r$,

$$E_n(f)_{w,p} = \mathcal{O}(n^{-\alpha}) \Leftrightarrow \Omega_\varphi^r(f, t)_{w,p} = \mathcal{O}(t^\alpha)$$

with $\varphi(x) = \sqrt{1-x^2}$. Hence, we obtain from the Proposition given above:

Theorem 4. *Suppose* $1 \leqslant p \leqslant \infty$, $\alpha > 0$, *and* s *is an integer satisfying* $s < \alpha \leqslant s+1$. *Then*

$$E_n(f)_{w,p} = \mathcal{O}(n^{-\alpha})$$

if and only if $f^{(s-1)} \in AC_{\mathrm{loc}}$ *and*

$$\left\| \left(\cos\frac{x}{2}\right)^{2\gamma_1 + s + 1/p} \left(\sin\frac{x}{2}\right)^{2\gamma_2 + s + 1/p} \Delta_h^2 f^{(s)}(\cos x) \right\|_{L_p(2h, \pi - 2h)} = \mathcal{O}(h^{\alpha - s}).$$

Here and in the sequel, differentiation has priority over substitution and substitution has priority over forming second difference, e.g., $\Delta_h^2 f^{(s)}(\cos x)$ denotes the second difference of the function $f^{(s)} \circ \cos$ at x.

The proof of Theorem 3 shows that Theorem 4 is actually true for γ_1, $\gamma_2 > -1/p$ when $1 \leqslant p < \infty$.

2. Let $(a, b) = (0, \infty)$, $\varphi(x) = \sqrt{x}$, and $w(z) = x^{\gamma_1}(1+x)^{\gamma_2}$ where $0 \leqslant \gamma_1 < 1 - 1/p$. The Szász-Kantorovich operators are defined by

$$S_n^* f(x) = \sum_{k=0}^{\infty} \left(n \int_{k/n}^{(k+1)/n} f(u)\, du \right) e^{-nx} \frac{(nx)^k}{k!}.$$

In [4, Theorem 10.1.3], we proved that for $1 \leqslant p < \infty$ and $0 < \alpha < 1$,

$$\| w(S_n^* f - f) \|_{L_p(0, \infty)} = \mathcal{O}(n^{-\alpha}) \tag{7.2}$$

if and only if

$$\Omega_\varphi^2(f, t)_{w, p} = \mathcal{O}(t^{2\alpha}). \tag{7.3}$$

We may now use the Ω-version of Theorem 1 (cf. the proof of Theorem 3) or (5.7) to obtain the following result.

THEOREM 5. *If* $1/2 - 1/p < \gamma_1 < 1 - 1/p$ *and* $0 < \alpha < 1$, *then* (7.2) *is equivalent to*

$$\| x^{2\gamma_1}(1+x^2)^{\gamma_2} x^{1/p} \Delta_h^2 f(x^2) \|_{L_p(2h, \infty)} = \mathcal{O}(h^{2\alpha}).$$

One can observe that (7.2) and (7.3) are equivalent even if we assume only $\gamma_1 > -1/p$, and then the proof of Theorem 3 shows the validity of Theorem 5 even if $1/2 - 1/p < \gamma_1$.

For example, if $0 < \alpha < 1$ and $p < 2$, then

$$\| S_n^* f - f \|_{L_p(0, \infty)} = \mathcal{O}(n^{-\alpha}) \tag{7.4}$$

if and only if

$$\| x^{1/p} \Delta_h^2 f(x^2) \|_{L_p(2h, \infty)} = \mathcal{O}(h^{2\alpha}). \tag{7.5}$$

It is remarkable that the example of Section 4 shows that (7.4) and (7.5) are not equivalent (for some α) when $p > 2$ (cf. (7.2)–(7.3)).

3. The Baskakov-Kantorovich operators are given by

$$V_n^* f(x) = \sum_{k=0}^{\infty} \left(n \int_{k/n}^{(k+1)/n} f(u)\, du \right) \binom{n+k-1}{k} x^k (1+x)^{-n-k}, \qquad (x \geqslant 0).$$

The corresponding quantities are $(a, b) = (0, \infty)$ and $\varphi(x) = \sqrt{x(1+x)}$. The results of this paper and [4, Theorem 10.1.3] imply for $p < 2$ and $0 < \alpha < 1$, that

$$\| V_n^* f - f \|_{L_p(0, \infty)} = \mathcal{O}(n^{-\alpha})$$

if and only if (see also Remark 2 in Section 3)

$$\|x^{2/p}e^{x/p}\Delta_h^2 f(x^2e^x)\|_{L_p(2h,\infty)} = \mathcal{O}(h^{2\alpha}).$$

Using Section 4, we note that the above is not valid (for some α) when $p > 2$.

4. Finally, let us consider $(a, b) = (0, \infty)$, $\varphi(x) = x$, $w(x) = x^{\gamma_1}(1+x)^{\gamma_2}$, where $-\infty < \gamma_1, \gamma_2 < \infty$, and $1 \leqslant p < \infty$. These are connected with the Post–Widder inversion formula for the Laplace transform or with the closely related Gamma operators given by

$$G_n f(x) = \frac{x^{n+1}}{n!} \int_0^\infty e^{-ux} u^n f\left(\frac{n}{u}\right) du.$$

For any $0 < \alpha \leqslant 1$ and $1 \leqslant p < \infty$, we proved in [4, Theorem 10.1.4] that

$$\|w(G_n f - f)\|_{L_p(0,\infty)} = \mathcal{O}(n^{-\alpha}) \Leftrightarrow \omega_\varphi^2(f, t)_{w,p} = \mathcal{O}(t^{2\alpha}).$$

Theorem 1 now implies:

THEOREM 6. *For every* $0 < \alpha \leqslant 1$ *and* $1 \leqslant p < \infty$,

$$\|w(G_n f - f)\|_{L_p(0,\infty)} = \mathcal{O}(n^{-\alpha})$$

if and only if

$$\|e^{\gamma_1 x}(1+e^x)^{\gamma_2}\, e^{x/p}\Delta_h^2 f(e^x)\|_{L_p(-\infty,\infty)} = \mathcal{O}(h^{2\alpha}).$$

We can look upon Theorem 6 as a constructive characterization of weighted Lipschitz spaces with weights

$$w(x) = e^{\rho x} + e^{\tau x}, \qquad \rho, \tau \in \mathbb{R}.$$

Many other applications can be given in connection with combinations of operators (see [4, Chap. 9]) but we do not wish to get into details.

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